Abelian groups with a p^2-bounded subgroup, revisited

By

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Abstract. Let Λ be a commutative local uniserial ring of length n, p a generator of the maximal ideal, and k the radical factor field. The pairs (B, A) where B is a finitely generated Λ-module and $A \subseteq B$ a submodule of B such that $p^m A = 0$ form the objects in the category $\mathcal{S}_m(\Lambda)$. We show that in case $m = 2$ the categories $\mathcal{S}_m(\Lambda)$ are in fact quite similar to each other: If also Δ is a commutative local uniserial ring of length n and with radical factor field k, then the categories $\mathcal{S}_2(\Lambda)/\mathcal{N}_\Lambda$ and $\mathcal{S}_2(\Delta)/\mathcal{N}_\Delta$ are equivalent for certain nilpotent categorical ideals $\mathcal{N}_\Lambda$ and $\mathcal{N}_\Delta$. As an application, we recover the known classification of all pairs (B, A) where B is a finitely generated abelian group and $A \subseteq B$ a subgroup of B which is p^2-bounded for a given prime number p.

1. History and Introduction

Let Λ be a commutative local uniserial ring of length n with radical generator p and radical factor field $k = \Lambda/p$. We consider pairs $(B; A)$ where B is a finitely generated Λ-module and A a submodule of B. Such pairs form the objects in the category $\mathcal{S}(\Lambda)$; a morphism from $(B; A)$ to $(D; C)$ is given by a map $f : B \to D$ which satisfies $f(A) \subset C$. We are particularly interested in the full subcategories $\mathcal{S}_m(\Lambda) \subset \mathcal{S}(\Lambda)$ (for $m \leq n$ a natural number) which consist of those pairs $(B; A)$ that satisfy $p^m A = 0$. For example if $\Lambda = \mathbb{Z}/p^n$ then we are dealing with pairs $(B; A)$ where B is a finite abelian p^n-bounded group and $A \subseteq B$ a subgroup satisfying $p^m A = 0$.

Each category $\mathcal{S}_m(\Lambda)$ has the Krull-Remak-Schmidt property, so every object has a unique direct sum decomposition into indecomposable ones. Examples for indecomposable objects are *pickets* which are pairs $(B; A)$ where the Λ-module B itself is indecomposable, hence cyclic.

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Since Λ is uniserial, each picket $(B;A)$ is determined uniquely by the lengths ℓ and m of the Λ-modules B and A; we write $P_m^\ell = (B;A)$.

Clearly, the complexity of categories of type $\mathcal{S}_m(\Lambda)$ increases with m. The categories $\mathcal{S}_0(\Lambda)$ and modΛ are equivalent so the only indecomposable objects in $\mathcal{S}_0(\Lambda)$ are the pickets of type P_0^ℓ. In the category $\mathcal{S}_1(\Lambda)$ (which we consider briefly in Section 3), every indecomposable object is a picket of type P_0^ℓ or P_1^ℓ. The category $\mathcal{S}_2(\Lambda)$ contains additional indecomposables which are not pickets; it turns out that an invariant which has been introduced by Prüfer [3, §7] in 1923 provides an efficient classification.

Definition. Let B be a Λ-module and $a \in B$ a nonzero element. The *height exponent* of a is

$$h_B(a) \;=\; h(a) \;=\; \max\{n \in \mathbb{N}_0 \;:\; a = p^n b \text{ for some } b \in B\};$$

the *height sequence* $H_B(a) = (h(a), h(pa), \ldots, h(p^\ell a))$ consists of the height exponents of the nonzero p-power multiples of a.

Example 1.1. For pickets, the height sequence consists of consecutive numbers: In a picket $(B;A) = P_m^\ell$ where $m > 0$, any generator for A has the height sequence $(\ell - m, \ell - m + 1, \ldots, \ell - 1)$.

Example 1.2. If $m \geq 2$ then there are height sequences which cannot be realized by pickets. The sequence $(s-1, t-1)$ where $s < t-1$ is realized by the pair $Q_s^t = (B;A)$ where $B = \Lambda/(p^t) \oplus \Lambda/(p^s)$ and $A = (p^{t-2}, p^{s-1})\Lambda$.

All pickets and all indecomposables of type Q_s^t have the property that the subgroup is either zero or cyclic. This is always the case for indecomposable objects in $\mathcal{S}_2(\Lambda)$ according to [2, Theorem 4]:

THEOREM 1.3. *Each pair $(B, A) \in \mathcal{S}_2(\mathbb{Z}/p^n)$ is a direct sum of indecomposable pairs; if (B, A) is an indecomposable pair then A is either zero or cyclic.*

A description of the indecomposable objects in terms of standard forms of matrices is given in [1, Theorem 7.5]. It turns out that whenever the pair $(B;A)$ is indecomposable with A nonzero, then the height sequence $H_B(a)$ of a subgroup generator a uniquely determines the isomorphism type of the given pair. Since there are $\frac{1}{2}(n^2 + n)$ height sequences of length at most 2 with values at most n, and since there are n isomorphism types of indecomposable pairs $(B;A)$ where $A = 0$, we deduce that there are in total $n + \frac{1}{2}(n^2 + n) = \frac{1}{2}(n^2 + 3n)$ indecomposable objects in $\mathcal{S}_2(\Lambda)$, up to isomorphism.

In this manuscript we recover the list of indecomposable objects in $\mathcal{S}_2(\Lambda)$ using poset representations, we demonstrate that the list does not only not depend on the choice of the base ring Λ, but that in fact all the categories of type $\mathcal{S}_2(\Lambda)$ are related:

Let Δ be a second commutative local uniserial ring such that Λ and Δ have the same length n and isomorphic radical factor fields. Clearly, the categories $\mathcal{S}_2(\Lambda)$ and $\mathcal{S}_2(\Delta)$ cannot be equivalent unless the rings Λ and Δ are isomorphic.

We define categorical ideals $\mathcal{N}_\Lambda \subset \mathcal{S}_2(\Lambda)$ and $\mathcal{N}_\Delta \subset \mathcal{S}_2(\Delta)$ which are "large enough" to make the factor categories equivalent,

$$\mathcal{S}_2(\Lambda)/\mathcal{N}_\Lambda \;\simeq\; \mathcal{S}_2(\Delta)/\mathcal{N}_\Delta,$$

and "small enough" so that the categories $\mathcal{S}_2(\Lambda)/\mathcal{N}_\Lambda$ and $\mathcal{S}_2(\Lambda)$ have the same indecomposable objects in the sense that no nonzero object in $\mathcal{S}_2(\Lambda)$ is isomorphic to zero when considered as object in $\mathcal{S}_2(\Lambda)/\mathcal{N}_\Lambda$.

We would like to emphasize that only basic methods from linear algebra are needed to establish the well-known list of the indecomposable representations of this poset, and hence to obtain the list of the indecomposable objects in $\mathcal{S}_2(\Lambda)$.

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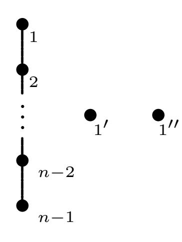
2. Poset Representations

We introduce the poset $\mathcal{P}_n$, define a functor $F : \mathcal{S}_2(\Lambda) \to \operatorname{rep}_k \mathcal{P}_n$ into the category of k-linear representations of $\mathcal{P}_n$, and show that the representations of type $F(P_m^\ell)$ and $F(Q_s^t)$ form a full list of the indecomposable representations in the image of F.

Let $\mathcal{P}_n$ be the following poset:

$\mathcal{P}_n :$



Recall that a representation $(V^0; (V^i)_{i\in\mathcal{P}_n})$ for $\mathcal{P}_n$ is a k-vector space V^0, the *total space* of V, together with subspaces $V^i \subset V^0$ for $i \in \mathcal{P}_n$, such that $V^i \subset V^j$ holds whenever $i < j$ in $\mathcal{P}_n$. For short we write V' for $V^{1'}$ and V'' for $V^{1''}$.

The category $\operatorname{rep}_k \mathcal{P}_n$ is a Krull-Remak-Schmidt category, so every representation has a unique direct sum decomposition into indecomposable representations. The indecomposable objects in $\operatorname{rep}_k \mathcal{P}_n$ are in $1-1$-correspondence to those indecomposable representations of the Dynkin diagram $\mathbb{D}_{n+2}$ which have support in the central point, thus there are

$$\begin{aligned} \#\{\operatorname{ind}\operatorname{rep}\mathbb{D}_{n+2}\} - \#\{\operatorname{ind}\operatorname{rep}\mathbb{A}_{n-1}\} - 2\#\{\operatorname{ind}\operatorname{rep}\mathbb{A}_1\} &= \\ = \quad (n^2+3n+2) - \frac{1}{2}(n^2-n) - 2 \quad &= \quad \frac{1}{2}n^2 + \frac{7}{2}n \end{aligned}$$

indecomposables. They are as follows: The total space V^0 of an indecomposable representation V either has dimension 1 or dimension 2. If $\dim V^0 = 1$ then V is isomorphic to one of the representations $V_{\ell,\ell',\ell''}$, where $0 \le \ell < n$ and $0 \le \ell', \ell'' \le 1$, defined as follows.

$$V^i_{\ell,\ell',\ell''} = \begin{cases} k & \text{if } i \le \ell \\ 0 & \text{if } i > \ell \end{cases} \quad V'_{\ell,\ell',\ell''} = \begin{cases} k & \text{if } \ell' = 1 \\ 0 & \text{if } \ell' = 0 \end{cases} \quad V''_{\ell,\ell',\ell''} = \begin{cases} k & \text{if } \ell'' = 1 \\ 0 & \text{if } \ell'' = 0 \end{cases}$$

If $\dim V^0 = 2$ then V is isomorphic to one of the representations $W_{s,t}$ where $1 \le s < t < n$:

$$W^i_{s,t} = \begin{cases} k \oplus k & \text{if} \quad i \le s \\ \Delta & \text{if} \quad s < i \le t \\ 0 & \text{if} \quad i > t \end{cases} \qquad W'_{s,t} = k \oplus 0 \qquad W''_{s,t} = 0 \oplus k$$

where $\Delta = k(1,1) \subset k \oplus k$ is the diagonal.

Given a pair $(B;A) \in \mathcal{S}_2(\Lambda)$, we obtain a representation V of $\mathcal{P}_n$ as follows. Consider the filtration for B given by the subspace $\operatorname{rad} A$:

$$\begin{array}{llll} L_0 & = A^- & = \operatorname{rad} A & \\ L_1 & = A^+ & = p^{-1}\operatorname{rad} A & = A + \operatorname{soc} B \\ L_2 & & = p^{-2}\operatorname{rad} A & \\ & & \vdots & \\ L_{n-1} & & = p^{1-n}\operatorname{rad} A & \\ L_n & & = p^{-n}\operatorname{rad} A & = B \end{array}$$

Here, as usual in this manuscript, we write p for the endomorphism of B given by multiplication by p. Thus, for a submodule $U \subset B$ we denote by pU and $p^{-1}U$ the image and the inverse image of U under this map. Note that subsequent quotients of the filtration for B are vector spaces; in particular, $V^0 = A^+/A^-$ will be the total space. For $\ell > 0$, the multiplication by p^ℓ defines maps $p^\ell : L_{\ell+1} \to A^+$, $p^\ell : L_\ell \to A^-$ which have image $\operatorname{rad}^\ell B \cap A^+$ and $\operatorname{rad}^\ell B \cap A^-$, respectively, and which

give rise to isomorphisms

$$\frac{L_{\ell+1}}{L_\ell} \xrightarrow{\cong} \frac{\operatorname{rad}^\ell B \cap A^+}{\operatorname{rad}^\ell B \cap A^-} \xrightarrow{\cong} \frac{(\operatorname{rad}^\ell B \cap A^+) + A^-}{A^-}$$

into the submodule $V^\ell = ((\operatorname{rad}^\ell B \cap A^+) + A^-)/A^-$ of V^0. We also set $V' = \operatorname{soc} B/A^-$ and $V'' = A/A^-$.

Note that all the spaces V^j have the form $\widetilde{C} = ((C \cap A^+) + A^-)/A^- \subseteq A^+/A^-$ for a suitable submodule C of B. We collect some properties of this construction $C \mapsto \widetilde{C}$.

Lemma 2.1. *Let $(B; A)$ be a pair in $\mathcal{S}_2(\Lambda)$ and C, C' be submodules of B.*

(1) *If $C \subseteq C'$ then $\widetilde{C} \subseteq \widetilde{C'}$.*
(2) *Always, $\widetilde{C \cap C'} \subseteq \widetilde{C} \cap \widetilde{C'}$ holds. If $A^- \subseteq C$ or $A^- \subseteq C'$ then equality holds.*
(3) *Always, $\widetilde{C} + \widetilde{C'} \subseteq \widetilde{C + C'}$ holds. We have equality if $C \subseteq A^+$ or $C' \subseteq A^+$.*

Proof. (1) If $C \subseteq C'$ then $(C \cap A^+) + A^- \subseteq (C' \cap A^+) + A^-$ holds and the assertion follows.
(2) The inclusion $(C \cap C' \cap A^+) + A^- \subseteq [(C \cap A^+) + A^-] \cap [(C' \cap A^+) + A^-]$ holds always. If $A^- \subseteq C$ is given, then the right hand side in the inclusion simplifies to $(C \cap A^+) \cap [(C' \cap A^+) + A^-]$ and by the modular law to $(C \cap C' \cap A^+) + A^-$.
(3) Similarly, $(C \cap A^+) + (C' \cap A^+) + A^- \subseteq [(C + C') \cap A^+] + A^-$ holds always. If also $C \subseteq A^+$ holds, then the left hand side simplifies to $C + (C' \cap A^+) + A^-$ and by the modular law to $[(C + C') \cap A^+] + A^-$. $\square$

As a consequence we obtain:

Proposition 2.2. (1) *The assignment which maps an object (B, A) in $\mathcal{S}_2(\Lambda)$ to the representation $V = (V^0, (V^j))$ of $\mathcal{P}_n$ given by*

$$V^0 = \widetilde{B};\ V^\ell = \widetilde{\operatorname{rad}^\ell B},\ \textit{for } 1 \le \ell \le n-1, \quad V' = \widetilde{\operatorname{soc} B}, \quad V'' = \widetilde{A},$$

defines an additive functor $F : \mathcal{S}_2(\Lambda) \to \operatorname{rep}_k \mathcal{P}_n$.
(2) *Each representation $V = F(B; A)$ satisfies the following conditions:*

$$V^{n-1} \subset V' \qquad \textit{and} \qquad V' + V'' = V^0.$$

Proof. It follows from Lemma 2.1 that V is a representation of $\mathcal{P}_n$ satisfying (2). If $f : (B; A) \to (D; C)$ is a morphism in $\mathcal{S}_2(\Lambda)$, then f maps the submodules of B, $A^- = \operatorname{rad} A$, A, $\operatorname{rad}^\ell B$ (where $0 \le \ell \le n-1$), $\operatorname{soc} B$, and A^+ into the corresponding submodules of D and

hence gives rise to a map $F(f)$ between the representations $F(B;A)$ and $F(D;C)$. $\square$

Definition. We denote by $\operatorname{rep}'_k \mathcal{P}_n$ the full subcategory of $\operatorname{rep}_k \mathcal{P}_n$ consisting of those representations which satisfy the condition (2).

Among the $\frac{1}{2}n^2 + \frac{7}{2}n$ indecomposable representations for $\mathcal{P}_n$, the two representations $V_{n-1,0,0}$ and $V_{n-1,0,1}$, and the $n-1$ representations $W_{s,n-1}$ where $0 \leq s \leq n-2$, do not have the property that $V^{n-1} \subset V'$. The condition that $V' + V'' = V^0$ excludes the n representations $V_{\ell,0,0}$ where $0 \leq \ell \leq n-1$. It turns out that all the remaining $\frac{1}{2}n^2 + \frac{3}{2}n$ indecomposable representations in $\operatorname{rep}'_k \mathcal{P}_n$ are in bijection with the indecomposable pairs in $\mathcal{S}_2(\Lambda)$ of type P_m^ℓ and Q_s^t:

Proposition 2.3. *The functor F gives rise to the following correspondence between pairs in $\mathcal{S}_2(\Lambda)$ and representations in $\operatorname{rep}_k \mathcal{P}_n$.*

$$\begin{array}{lll} F(P_0^\ell) & = V_{\ell-1,1,0} & (1 \leq \ell \leq n) \\ F(P_1^\ell) & = V_{\ell-1,1,1} & (1 \leq \ell \leq n) \\ F(P_2^\ell) & = V_{\ell-2,0,1} & (2 \leq \ell \leq n) \\ F(Q_s^t) & = W_{s-1,t-2} & (0 \leq s-1 < t-2 \leq n-2) \end{array}$$

As a consequence, $F : \mathcal{S}_2(\Lambda) \to \operatorname{rep}'_k \mathcal{P}_n$ is a dense functor. Moreover, all pairs of type P_m^ℓ and Q_s^t are indecomposable and pairwise nonisomorphic. $\square$

We will see in Section 4 that F is full. Hence the pairs P_m^ℓ and Q_s^t form a full list of the indecomposable objects in $\mathcal{S}_2(\Lambda)$.

3. Picket Decomposition

We show in this section that every object in $\mathcal{S}_1(\Lambda)$ is direct sum of pickets of type P_0^ℓ or P_1^ℓ where $1 \leq \ell \leq n$. We deduce in Theorem 3.4 that every pair $(B;A) \in \mathcal{S}_2(\Lambda)$ with the additional property that $\operatorname{soc} B \subset A$ is a direct sum of pickets of type P_1^ℓ or P_2^ℓ.

The ring Λ is selfinjective of Loewy length n. It turns out that the pair $P_0^n = (\Lambda; 0)$ is a relatively injective indecomposable object in $\mathcal{S}(\Lambda)$ with source map the inclusion $u_n : P_0^n \to P_1^n$. We give a direct proof of this result which follows also from [4, Proposition 1.4].

Lemma 3.1. *Let (B, A) be an object in $\mathcal{S}(\Lambda)$ and $f : P_0^n \to (B, A)$ a morphism. Either f is a split monomorphism or else f factors over the inclusion $u_n : P_0^n \to P_1^n$. Thus, P_0^n is relatively injective in the sense that every monomorphism from P_0^n with cokernel in $\mathcal{S}(\Lambda)$ is a split monomorphism.*

Proof. Suppose a morphism $P_0^n \to (B,A)$ is given by a map $f:\Lambda\to B$. Clearly, if f is not a monomorphism or if $\operatorname{Im} f\cap A\neq 0$, then the map $P_0^n \to (B;A)$ factors over u_n. It remains to deal with the case that f is a monomorphism such that $E\cap A=0$ where $E=\operatorname{Im} f$. Let $e_A : A\to E(A)$ and $e_B: B\to E(B)$ be injective envelopes, then there is a map h which makes the following diagram commutative.

$$\begin{array}{ccc} E\oplus A & \xrightarrow{\text{incl}} & B \\ {\scriptstyle 1\oplus e_A}\downarrow & & \downarrow{\scriptstyle e_B} \\ E\oplus E(A) & \xrightarrow{h} & E(B) \end{array}$$

Since $E\oplus A$ is large in $E\oplus E(A)$ and since the composition $h\circ(1\oplus e_A)$ is a monomorphism, also h is a monomorphism. Then h is a split monomorphism, so we can write $E(B)=E\oplus E(A)\oplus E''$ and define $B'=\{b\in B: e_B(b)\in E(A)\oplus E''\}$. Since $E\subset B$, the assertions $B=E\oplus B'$ and $A\subseteq B'$ hold. The decomposition $(B,A)=(E,0)\oplus(B',A)$ demonstrates that $P_0^n\to(B;A)$ is a split monomorphism.
Suppose a monomorphism $P_0^n \to (B;A)$ is such that the cokernel $(D;C)$ is an object in $\mathcal{S}(\Lambda)$. Then we have a commutative diagram with exact rows in which the vertical maps are monomorphisms.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \text{incl}} & & \downarrow{\scriptstyle \text{incl}} & & \\ 0 & \longrightarrow & \Lambda & \xrightarrow{f} & B & \longrightarrow & D & \longrightarrow & 0 \end{array}$$

By the snake lemma, the composition $\Lambda\xrightarrow{f}B\to B/A$ is a monomorphism, hence $\operatorname{Im} f\cap A=0$ and we have just seen that this implies that $P_0^n\to(B;A)$ is a split monomorphism. □

LEMMA 3.2. *The object P_1^n is injective in $\mathcal{S}_1(\Lambda)$.*

Proof. Let $P_1^n\to(B;A)$ be a monomorphism in $\mathcal{S}_1(\Lambda)$. If the image is $(E,\operatorname{soc} E)$ then the injective Λ-module E has a complement in B, say $B=E\oplus B'$. Since A is semisimple and $\operatorname{soc} E\subseteq A$ we obtain $A\cap(E\oplus B')=A\cap(\operatorname{soc} E\oplus\operatorname{soc} B')=\operatorname{soc} E\oplus(A\cap\operatorname{soc} B')=\operatorname{soc} E\oplus(A\cap B')$ by the modular law and hence the pair $(B;A)$ decomposes as $(E;\operatorname{soc} E)\oplus(B';A\cap B')$. □

PROPOSITION 3.3. *Every object in $\mathcal{S}_1(\Lambda)$ is a direct sum of objects of type P_0^ℓ or P_1^ℓ where $1\le\ell\le n$.*

Proof. Assume that the pair $(B;A)\in\mathcal{S}_1(\Lambda)$ is nonzero. We show that $(B;A)$ has a summand of type P_0^ℓ or P_1^ℓ where ℓ is the Loewy length of

B. Indeed B, having Loewy length ℓ, has an indecomposable summand, say E, which is a projective-injective $\Lambda/(p^\ell)$-module. According to Lemma 3.1, either $P_0^\ell = (E;0)$ splits off as a direct summand of $(B;A)$, or else $P_1^\ell = (E;\operatorname{soc} E)$ can be embedded into $(B;A)$. In the second case, P_1^ℓ splits off as a direct summand since it is an injective module (Lemma 3.2). □

Theorem 3.4. *Every pair $(B;A)$ in $\mathcal{S}_2(\Lambda)$ with the extra property that $\operatorname{soc} B \subset A$, is isomorphic to a direct sum of pickets P_m^ℓ where $m = 1$ or 2 and $m \le \ell \le n$.*

Proof. For each pair $(B;A) \in \mathcal{S}_2(\Lambda)$ which satisfies $\operatorname{soc} B \subset A$ consider the picket decomposition of the corresponding pair $(B;\operatorname{rad} A)$ in $\mathcal{S}_1(\Lambda)$ given by Proposition 3.3,

$$(B;\operatorname{rad} A) \;=\; \bigoplus_{i=1}^{s} (B_i;\operatorname{rad} A \cap B_i).$$

We show that this yields the picket decomposition for $(B;A)$:

$$(B;A) \;=\; \bigoplus_{i=1}^{s} (B_i; A \cap B_i).$$

Consider p as an endomorphism of B and write pU and $p^{-1}U$ for the image and the inverse image of the submodule $U \subset B$.

$$\begin{aligned}
A &\overset{(1)}{=} A + \operatorname{soc} B \\
&= p^{-1}pA \\
&\overset{3.3}{=} p^{-1}\Big(\sum_i pA \cap B_i\Big) \\
&\overset{(2)}{=} \sum_i p^{-1}\left(pA \cap B_i\right) \\
&= \sum_i \left(p^{-1}pA \cap p^{-1}B_i\right) \\
&\overset{(1)}{=} \sum_i \left(A \cap (B_i + \operatorname{soc} B)\right) \\
&\overset{(1)}{=} \operatorname{soc} B + \sum_i \left(A \cap B_i\right) \\
&= \sum_i \left(\operatorname{soc} B_i + A \cap B_i\right) \\
&= \bigoplus_i A \cap B_i
\end{aligned}$$

In the equations labelled (1), equality holds since $\operatorname{soc} B \subset A$, and equality in (2) follows from $pA \cap B_i \subset pB$. $\square$

4. Homomorphism Categories

We show in Theorem 4.3 that the functor $F : \mathcal{S}_2(\Lambda) \to \operatorname{rep}_k \mathcal{P}_n$ is full.

Definition. Let $\mathcal{C}$ be any category. The *homomorphism category* $\mathcal{H}(\mathcal{C})$ has as objects all triples $(B; A; f)$ or $(B \xleftarrow{f} A)$ where $f : A \to B$ is a morphism in $\mathcal{C}$. A morphism in $\mathcal{H}(\mathcal{C})$ from $(B; A; f)$ to $(D; C; e)$ is a pair $(h : B \to D, g : A \to C)$ of morphisms in $\mathcal{C}$ which satisfies $eg = hf$. If $\mathcal{C}$ is an abelian category, then so is $\mathcal{H}(\mathcal{C})$; in this case kernels and cokernels, and hence pull-backs and push-outs, are computed componentwise. For short we write $\mathcal{H}(\Lambda)$ for $\mathcal{H}(\operatorname{mod}\Lambda)$, thus $\mathcal{S}(\Lambda) \subset \mathcal{H}(\Lambda)$ is an embedding of a full subcategory.

Proposition 4.1. *Given a pair* $(B; A^+) \in \mathcal{S}_2(\Lambda)$ *where* $\operatorname{soc} B \subset A^+$, *and a subspace* U *of* A^+/A^- *where* $A^- = \operatorname{rad} A^+$, *then there is a unique*

submodule A of B such that the diagram

$$(*)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & (B;A^-) & \longrightarrow & (B;A) & \longrightarrow & (0;U) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (B;A^-) & \longrightarrow & (B;A^+) & \longrightarrow & (0;A^+/A^-) & \longrightarrow & 0 \end{array}$$

is a pull-back diagram in the category $\mathcal{H}(\Lambda)$. Conversely, every pair $(B;A)$ arises in this way.

Proof. The bottom sequence in $(*)$ is a short exact sequence in the category $\mathcal{H}(\Lambda)$, and the pull-back along the inclusion $U \to A^+/A^-$ yields an object $X \in \mathcal{H}(\Lambda)$ and two monomorphisms $(B;A^-) \to X \xrightarrow{v} (B;A^+)$. Thus, X is in $\mathcal{S}_2(\Lambda)$ and by identifying X with $\operatorname{Im} v$ we obtain a uniquely determined submodule $A \subset B$ such that the above diagram $(*)$ is commutative and has exact rows.
In order to realize a pair $(B;A) \in \mathcal{S}_2(\Lambda)$ as a pull-back, take $A^+ = A + \operatorname{soc} B$, $A^- = \operatorname{rad} A$ and for U the subspace A/A^- of A^+/A^-. Then the diagram $(*)$ is commutative with exact rows and hence is a pull-back diagram since the vertical map on the left is an isomorphism. $\square$

Note that the last term A^+/A^- in the bottom sequence in $(*)$ is the total space of the poset representation $F(B;A^+)$. Our next result shows that any morphism $F(B;A^+) \to F(D;C^+)$ between poset representations can be lifted to a map $(B;A^+) \to (D;C^+)$:

Proposition 4.2. *Suppose that the pairs $(B;A)$ and $(D;C)$ in $\mathcal{S}_2(\Lambda)$ satisfy* $\operatorname{soc} B \subset A$ *and* $\operatorname{soc} D \subset C$. *Then the functor F induces an isomorphism*

$$\frac{\operatorname{Hom}_{\mathcal{S}_2(\Lambda)}((B;A),(D;C))}{\mathcal{N}((B;A),(D;C))} \xrightarrow{\cong} \operatorname{Hom}_{\operatorname{rep}_k \mathcal{P}_n}(F(B;A),F(D;C))$$

where $\mathcal{N}((B;A),(D;C))$ consists of all maps $f:(B;A) \to (D;C)$ such that $f(A) \subset \operatorname{rad} C$.

Proof. According to Theorem 3.4, the pairs $(B;A)$ and $(D;C)$ are direct sums of pickets of type P_1^s or P_2^s. Since F is an additive functor, we may assume that both $(B;A)$ and $(D;C)$ are in fact such pickets.
The kernel of the map $F_{(B;A),(D;C)}$ is $\mathcal{N}((B;A),(D;C))$, so it remains to show that any nonzero map $h: F(B;A) \to F(D;C)$ lifts to a map

$f : (B; A) \to (D; C)$ which makes the following diagram commutative.

$$\begin{tikzcd}
0 \arrow[r] & (B;A^-) \arrow[r] \arrow[d, "f^-"'] & (B;A) \arrow[r] \arrow[d, "f"'] & (0;A/A^-) \arrow[r] \arrow[d, "(0;h^0)"] & 0 \\
0 \arrow[r] & (D;C^-) \arrow[r] & (D;C) \arrow[r] & (0;C/C^-) \arrow[r] & 0
\end{tikzcd}$$

Since $\operatorname{soc} B \subset A$, the space $F(B;A)'' = A/A^-$ is nonzero and hence $F(B;A)$ is a poset representation of type $V_{\ell,\ell',1}$. Also $F(D;C)$ must be isomorphic to a representation of type $V_{m,m',1}$. If the space $V'_{m,m',1}$ is zero (that is, if $m' = 0$), then also the space $V'_{\ell,\ell',1}$ must be zero. We obtain that the length of A, which is $L = 2 - \ell'$, is at least the length $2-m'$ of C. Using the projectivity of A as a $\Lambda/(p^L)$-module, we obtain a lifting $g : A \to C$ for h^0:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "g"'] & A/A^- \arrow[d, "h^0"] \\
C \arrow[r] & C/C^-
\end{tikzcd}$$

Similarly, whenever a space $V^i_{m,m',1}$ is zero, then so is the space $V^i_{\ell,\ell',1}$, and hence $\ell \le m$ holds. Let $D \to E$ be the inclusion into an injective envelope. Then we can extend the composition $A \stackrel{g}{\to} C \stackrel{\text{incl}}{\to} D \to E$ to a map $e : B \to E$. Since $\ell \le m$, the image of e is contained in D so $f = e|_{B,D}$ is an extension of g which makes the following diagram commutative:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "g"'] & B \arrow[d, "f"] \\
C \arrow[r] & D
\end{tikzcd}$$

This extension satisfies $F(f) = h$, finishing the proof. $\square$

THEOREM 4.3. *The functor $F : \mathcal{S}_2(\Lambda) \to \operatorname{rep}_k \mathcal{P}_n$ is full.*

Proof. Given two objects $(B;A)$ and $(D;C)$ in $\mathcal{S}_2(\Lambda)$, put $U = F(B;A)$ and $V = F(D;C)$ and let $h : U \to V$ be a morphism in the category $\operatorname{rep}_k \mathcal{P}_n$. We show that there is a morphism $f : (B;A) \to (D;C)$ in $\mathcal{S}_2(\Lambda)$ such that $F(f) = h$.

Note that the representations $F(B;A)$ and $F(B;A^+)$ coincide in all positions, only $F(B;A)''$ may be a proper subspace of $F(B;A^+)''$; in fact, $F(B;A^+)''$ coincides with the total space $F(B;A^+)^0 = F(B;A)^0$. Thus, h defines a morphism $F(B;A^+) \to F(D;C^+)$. By Proposition 4.2, this

morphism lifts to a map $f^+ : (B; A^+) \to (D; C^+)$ which makes the diagram in $\mathcal{H}(\Lambda)$ commutative.

$$\begin{tikzcd}
0 \arrow[r] & (B;A^-) \arrow[r] \arrow[d, "f^-"'] & (B;A^+) \arrow[r] \arrow[d, "f^+"'] & (0;U^0) \arrow[r] \arrow[d, "(0,h^0)"] & 0 \\
0 \arrow[r] & (D;C^-) \arrow[r] & (D;C^+) \arrow[r] & (0;V^0) \arrow[r] & 0
\end{tikzcd}$$

Here $f^- = f^+|_{(B;A^-),(D;C^-)}$ is the restriction of f^+. The morphism $h : F(B; A) \to F(D; C)$ also yields the following commutative diagram.

$$\begin{tikzcd}
(0;U'') \arrow[r, "(0;h'')"] \arrow[d] & (0;V'') \arrow[d] \\
(0;U^0) \arrow[r, "(0;h^0)"'] & (0;V^0)
\end{tikzcd}$$

We obtain the desired map f as a pull-back in the abelian category $\mathcal{H}(\mathcal{H}(\Lambda))$ which has as objects all homomorphisms in $\mathcal{H}(\Lambda)$. The two diagrams above form the following diagram in $\mathcal{H}(\mathcal{H}(\Lambda))$, which has an exact row.

$$\begin{tikzcd}
 & & & (0;h'') \arrow[d] & \\
0 \arrow[r] & f^- \arrow[r] & f^+ \arrow[r] & (0;h^0) \arrow[r] & 0
\end{tikzcd}$$

The pullback of this diagram,

$$(**) \qquad \begin{tikzcd}
0 \arrow[r] & f^- \arrow[r] \arrow[d, equal] & f \arrow[r] \arrow[d] & (0;h'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f^- \arrow[r] & f^+ \arrow[r] & (0;h^0) \arrow[r] & 0
\end{tikzcd}$$

yields an object f in $\mathcal{H}(\mathcal{H}(\Lambda))$.

Since pull-backs are computed componentwise, f is in fact a morphism between the two pull-backs $(B; A)$ and $(D; C)$ computed in the category $\mathcal{H}(\Lambda)$ (see Proposition 4.1). In other words, $f^+ : (B; A^+) \to (D; C^+)$ restricts to a map $f : (B; A) \to (D; C)$.

$$\begin{tikzcd}
(B;A^-) \arrow[r] \arrow[d, "f^-"'] & (B;A) \arrow[r] \arrow[d, "f"'] & (B;A^+) \arrow[d, "f^+"] \\
(D;C^-) \arrow[r] & (D;C) \arrow[r] & (D;C^+)
\end{tikzcd}$$

Since $\operatorname{rad} A = \operatorname{rad} A^+$ and $\operatorname{rad} C = \operatorname{rad} C^+$ we have that the representations $F(B; A)$ and $F(B; A^+)$, and also the representations $F(D; C)$ and $F(D; C^+)$, coincide in all positions except possibly at $1''$. Since f

is the restriction of f^+, Proposition 4.2 implies that the linear maps $F(f)^j$ and h^j coincide for all $j \in \mathcal{P}_n \backslash \{1''\}$. For the space at $1''$ consider the submodule component of the top row in $(**)$:

$$\begin{tikzcd} 0 \arrow[r] & A^- \arrow[r] \arrow[d, "f|_{A^-,C^-}"'] & A \arrow[r] \arrow[d, "f|_{A,C}"'] & U'' \arrow[r] \arrow[d, "h''"] & 0 \\ 0 \arrow[r] & C^- \arrow[r] & C \arrow[r] & V'' \arrow[r] & 0 \end{tikzcd}$$

Since $F(B;A)'' = A/A^- = U''$ and $F(D;C)'' = C/C^- = V''$ we obtain that $F(f)'' = h''$, finishing the proof that $F(f) = h$. □

5. Corollaries

We have seen in Proposition 2.3 and Theorem 4.3 that the functor $F : \mathcal{S}_2(\Lambda) \to \mathrm{rep}'_k \mathcal{P}_n$ is full and dense. We first consider the kernel.

For pairs $(B;A)$ and $(D;C)$ in $\mathcal{S}_2(\Lambda)$ define the following subgroup of $\mathrm{Hom}_{\mathcal{S}}((B;A),(D;C))$:

$$\mathcal{N}\big((B;A),(D;C)\big) \quad = \quad \big\{f : (B;A) \to (D;C) \;\big|\; f(A^+) \subset C^-\big\}.$$

This generalizes the definition given in Proposition 4.2. The collection $\mathcal{N}_\Lambda$ or $\mathcal{N}$ of all such subgroups forms a categorical ideal in $\mathcal{S}_2(\Lambda)$.

Lemma 5.1. *If the length of Λ is $n > 1$ then $\mathcal{N}$ has nilpotency index $n+1$.*

Note that if $n = 1$ then $\mathcal{N} = 0$.

Proof. Given objects $(B_i;A_i) \in \mathcal{S}_2(\Lambda)$ for $0 \le i \le n+1$, morphisms $f_i : (B_{i-1};A_{i-1}) \to (B_i;A_i)$ in $\mathcal{N}$ for $1 \le i \le n+1$, and an element $b \in B_0$. Then $p^{n-1}b \in \mathrm{soc}\, B_0$, hence $f_1(p^{n-1}b) \in \mathrm{rad}\, A_1$. For each i we have that if $f_i \cdots f_1(p^{n-i}b) \in \mathrm{rad}\, A_i$ then $f_i \cdots f_1(p^{n-i-1}b) \in A_i + \mathrm{soc}\, B_i$ and hence $f_{i+1} \cdots f_1(p^{n-i-1}b) \in \mathrm{rad}\, A_{i+1}$. Thus, $f_n \cdots f_1(b) \in \mathrm{rad}\, A_n$ and hence $f_{n+1} \cdots f_1(b) = 0$. Conversely, if $n > 1$ then the nilpotency index is not less than $n+1$ since the following composition of n maps is nonzero.

$$P_1^n \overset{\mathrm{incl}}{\to} P_2^n \overset{\cdot p}{\to} P_2^n \overset{\cdot p}{\to} \quad \cdots \quad \overset{\cdot p}{\to} P_2^n$$

□

Corollary 5.2. *The functor $F : \mathcal{S}_2(\Lambda) \to \mathrm{rep}_k \mathcal{P}_n$ induces an equivalence of categories*

$$\bar{F} : \mathcal{S}_2(\Lambda)/\mathcal{N} \to \mathrm{rep}'_k \mathcal{P}_n.$$

Proof. We show that the kernel of F is $\mathcal{N}$. Let $f : (B;A) \to (D;C)$ be a map in $\mathcal{S}_2(\Lambda)$ and denote by f^+ the map $f^+ = f : (B;A^+) \to (D;C^+)$. Since $\mathrm{rad}\, A = \mathrm{rad}\, A^+$ and $\mathrm{rad}\, C = \mathrm{rad}\, C^+$ hold, the linear

maps $F(f)^0$, $F(f^+)^0$ coincide. By Proposition 4.2, $F(f^+)^0 = 0$ if and only if $f^+ \in \mathcal{N}((B; A^+), (D; C^+))$. Then $f(A^+) \subset C^-$, but this is the condition for f to be in $\mathcal{N}((B; A), (D; C))$. $\square$

Since $\mathcal{N}$ is a nilpotent ideal, the canonical functor $\mathcal{S}_2(\Lambda) \to \mathcal{S}_2(\Lambda)/\mathcal{N}$ preserves indecomposable objects and reflects isomorphisms, so the isomorphism classes of objects in $\mathcal{S}_2(\Lambda)$ and $\mathcal{S}_2(\Lambda)/\mathcal{N}$ are in a natural bijection. As a consequence, if also Δ is a commutative local uniserial ring of length n and with radical factor field k, then the categories $\mathcal{S}_2(\Lambda)$ and $\mathcal{S}_2(\Delta)$ admit a "natural" bijection between their objects.

Corollary 5.3. *Suppose that Λ and Δ are commutative local uniserial rings of the same length n and with radical factor fields isomorphic to k. Then the following categories are equivalent:*

$$\mathcal{S}_2(\Lambda)/\mathcal{N}_\Lambda \quad \cong \quad \mathcal{S}_2(\Delta)/\mathcal{N}_\Delta \quad \square$$

For example, the categories $\mathcal{S}_2(\mathbb{Z}/(p^n))/\mathcal{N}$ and $\mathcal{S}_2(k[T]/(T^n))/\mathcal{N}$ are equivalent if $k = \mathbb{Z}/p$.

Question: We have seen that $\mathcal{N}$ is an ideal of nilpotency index $n+1$. Is there a pair $\mathcal{L}_\Lambda$, $\mathcal{L}_\Delta$, of ideals of nilpotency index n such that the above Corollary holds with $\mathcal{N}$ replaced by $\mathcal{L}$? For example, an ideal of nilpotency index n is given by all maps which factor through the multiplication by p. Note that we cannot expect to have an ideal $\mathcal{L}$ of nilpotency index $r < n$: The endomorphism ring of the pair $X = (\Lambda; 0)$ has filtration

$$0 = \mathcal{L}^r(X,X) \subset \mathcal{L}^{r-1}(X,X) \subset \cdots \subset \mathcal{L}(X,X) \subset \Lambda$$

in which not all subsequent factors can be semisimple. Hence the factor ring $\Lambda/\mathcal{L}(X,X)$ is not a field. In particular if $\Lambda = \mathbb{Z}/(p^n)$ and $\Delta = k[T]/(T^n)$ then the following two endomorphism rings are not isomorphic: $\mathrm{End}_{\mathcal{S}_2(\Lambda)/\mathcal{L}}(\Lambda, 0) \ncong \mathrm{End}_{\mathcal{S}_2(\Delta)/\mathcal{L}}(\Delta, 0)$.

We can also deal with the case where Λ is a PID and p a prime element. By primary decomposition, any pair $(B; A)$ where B is a finitely generated Λ-module and A a p^2-bounded submodule of B has a unique direct sum decomposition into pair of type $(F; 0)$ where F is a finitely generated free Λ-module, a pair of type $(D; 0)$ where p acts as automorphism on D, and a pair $(B; A)$ where B is p^n-bounded for some natural number n.

Corollary 5.4. *Let Λ be a principal ideal domain, p a prime element, B a finitely generated Λ-module and A a submodule of B which is p^2-bounded. Then the pair (B, A) has a direct sum decomposition,*

unique up to isomorphy and reordering, into finitely many indecomposable pairs (a) of type $(\Lambda/(q);0)$ *where q is a prime power relatively prime to p, or (b) of type* P_m^ℓ *or* Q_s^t *in* $\mathcal{S}_2(\Lambda/(p^n))$ *for some* $n \in \mathbb{N}$*, or (c) indecomposable projective of type* $(\Lambda;0)$ $\square$

Returning to Prüfer's height sequences, we see that an indecomposable pair is either given by a power of a prime ideal in Λ, or else by the height sequence of a subgroup generator.

COROLLARY 5.5. *Let Λ be a principal ideal domain, p a prime element, B a finitely generated Λ-module and A a submodule of B which is p^2-bounded. An indecomposable pair* $(B;A)$ *is*

(1) *either isomorphic to* $(\Lambda/Q;0)$ *for a uniquely determined power Q of a prime ideal, or else,*
(2) *if $A = a\Lambda$ is (nonzero) cyclic, determined uniquely, up to isomorphism, by the height sequence* $H_B(a)$.

Conversely, every power of a prime ideal, and every height sequence of length at most 2*, can be realized by an indecomposable pair.* $\square$

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